0.1 Introduction

This is a topic on the axioms of categories, metacategories and supercategories that are relevant, respectively, to mathematics and meta-mathematics. Lawvere's elementary theory of abstract categories (ETAC) provides an axiomatic construction of the theory of categories and functors. Intuitively, with this terminology and axioms, a category is meant to be any structure which is a direct interpretation of ETAC. A functor is then understood to be a triple consisting of two such categories and of a rule F ('the functor') which assigns to each arrow or morphism x of the first category, a unique morphism, written as 'F(x)' of the second category, in such a way that the usual two conditions on both objects and arrows in the standard functor definition are fulfilled –the functor is well behaved, i.e., it carries object identities to image object identities, and commutative diagrams to image commmutative diagrams of the corresponding image objects and image morphisms. At the next level, one then defines natural transformations or functorial morphisms between functors as meta-level abbreviated formulas and equations pertaining to commutative diagrams of the distinct images of two functors acting on both objects and morphisms. As the name indicates natural transformations are also well–behaved in terms of the ETAC equations that are satisfied by natural transformations.

0.2 ETAS and ETAC

Categories were defined in refs. [9, 10] as mathematical interpretations of the 'elementary theory of abstract categories' (ETAC). One can generalize the theory of categories to higher dimensions– as in higher dimensional algebra (HDA)– by defining multiple composition laws and allowing higher dimensional, functorial morphisms of several variables to be employed in such higher dimensional structures. Thus, one can introduce an elementary theory of supercategories (ETAS;([1, 2]) as a natural extension of Lawvere's ETAC theory to higher dimensions ([7]). Then, supercategories can be defined as mathematical interpretations of the ETAS axioms as in ref.[1].

Definition 0.1. A concrete *metagraph* $\mathcal{M}_G$ consists of objects, $A, B, C, ...$ and arrows $f, g, h, ...$ between objects, and two operations as follows:

- a *domain operation*, *dom*, which assigns to each arrow f an object $A = dom\ f$
- a *codomain operation*, *cod*, which assigns to each arrow f an object $B = cod\ f$, represented as $f : A \to B$ or $A \xrightarrow{f} B$

Remark 0.1. Related concepts to the general notion of a supercategory recalled above can also be rendered graphically on a computer as a multigraph or a *hypergraph*. More generally, the class of metagraphs can be also defined as a specific class of supercategories. On the other hand, a supercomputer architecture and operating system software are examples of realizations of relatively simple, or lower dimensional supercategories, as explained in further detail in the next subsections.

1 ETAS Axioms:

1. **(S1)**. All symbols, formulas and the eight axioms defined in ETAC are, respectively, also ETAS symbols, formulas and axioms; thus, for any letters x, y, i, u, A, B, and *unary function* symbols Δ_0 and Δ_1, and *composition laws* Γ_i, the following are defined as *formulas*: $\Delta_0(x) = A$,$\Delta_1(x) = B$, $\Gamma(x, y; u)$, and $x = y$.

   The above formulas are to be, respectively, interpreted as "A is the domain of x", "B is the codomain, or range, of x", "u is the composition x followed by y", and "x equals y"; letters $i, j, k, l, m, ...$ are to be interpreted as "either element, set or class (C) indices". An example of valid ETAC and ETAS formula is a couple or pair of two letters written as "(x, y)"; a more general related example is that of Cartesian or direct products $\Pi_{i \ in \ C}$.

2. **(S2)**. There are several composition laws defined in ETAS (as distinct from ETAC where there is only one composition law for each interpretation in any specific type of category); such multiple composition laws Γ_i, with $i \ in \ C$ are interpreted as "definitions of multiple (specific) mathematical structures, within the same supercategory §" .

In the case of general algebras, the multiple composition laws are interpreted as *"definitions of algebraic structures"*, (whereas *categorical algebra* is interpreted as being "defined by a single composition law $\Gamma_1 = \circ$ (or "*" for -involution or C^* -algebras)". An ETAC structure is thus identified by the singleton $\{1\}$ index set.

1.1 Examples of supercategories

pseudographs, hypergraphs, 1-categories, categorical algebras, 2-categories, n-categories, functor categories, super-categories, super-diagrams, functor supercatgeories, double groupoids, double categories, organismic supercategories, self-replicating quantum automata, standard Heyting topos, generalized LM_n-logic algebra topoi, double algebroids, super-categories of double algebroids, and any higher dimensional algebra (HDA) are examples of supercategories of various orders.

1.2 Graphic example of a supercategory

A pictorial representation of a particular class of metagraphs –the class of multigraphs, M_g– is also useful as a visual or 'geometric' (or topological) representation of a specific example of a supercategory defined over the topological space of the multigraph with the composition operations of the supercategory heteromorphisms defined, in this case of the multigraph, by the concatenations of the multigraph vertices in n dimensions for a finite , n-dimensional multigraph that can be graphically rendered on a computer.

1.3 Metagraphs and Metacategories

Definition 1.1. A more recent version of Lawvere's axioms was presented by MacLane (2000) in which a *metagraph* is first defined as a structure consisting of *objects* $a, b, c, ...x, y, z$, *arrows* $f, g, h, ...$, and two *operations*– the Domain (which assigns to each arrow f an object $a = dom\ f$), and a Codomain (which assigns to each arrow f an object $b = cod\ f$. Such operations can be readily represented by displaying f as an actual arrow $. \rightarrow .$ starting at the $dom\ f$ and ending at $cod\ f$, $f : a \rightarrow b$. With this pictorial, or 'geometric' representation, a finite number of arrows is depicted as a *finite graph*. Then, one defines a *metacategory* as a *metagraph* with two additional operations, *Identity* and *Composition*(viz. [12]). *Identity* assigns to each object a an arrow $id_a = 1_a : a \rightarrow a$. A *composition* operation assigns to each pair of arrows (f, g) with $dom\ g = cod\ f$ an arrow called their composite, $g \circ f : dom\ f \rightarrow cod\ g$.

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